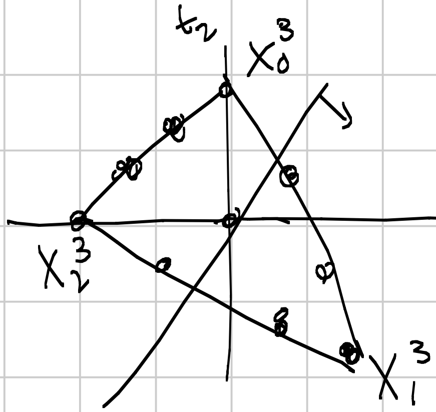
Kempf's theorem

Note Title 11/9/2016

Finishing HM criterion example:

Char wrt. $T \subset SL_3$, standard max torus



1PS $\rightsquigarrow$ codirection

limit $p_0 := \lim_{t \to 0} \lambda(t) \cdot p$.

$wt_\lambda(\mathcal{O}(1)_{p_0}) = -\langle \lambda, \chi_{min} \rangle$

Point $p \in \mathbb{P}(\mathrm{Sym}^3 \mathbb{C}^3)$ is:

i) T-semistable iff $St(p) \subset \mathfrak{t}_{\mathbb{R}}$ contains origin

ii) G-semistable iff $g \cdot p$ is T-semistable for all $g \in G$ (because of HM criterion)

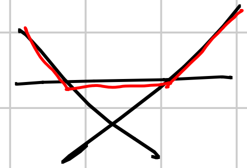
$\boxed{\text{More is true:}}$ based on a norm $|\lambda|$ on 1PS ($|\cdot|$ should be W-invariant), $N =$ Char of T

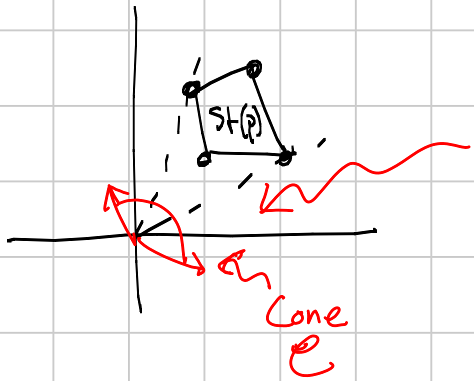
Given $St(p) \subset N_{\mathbb{R}}$, $\exists!\ \lambda$ which minimizes

$$\nu(p,\lambda) = \frac{1}{|\lambda|} wt(\mathcal{O}(1)_{p_0}) = -\langle \frac{1}{|\lambda|}\lambda, \chi_{min} \rangle$$

lowest λ-wt appearing in $St(p)$

This is because $\nu(p,\lambda) = \max_{\chi \in St(p)} \langle \frac{-\lambda}{|\lambda|}, \chi \rangle$

This is convex upward in λ: 



minimize $\max_{\chi \in St(p)} \left\langle \frac{-\lambda}{|\lambda|}, \chi \right\rangle$ over rational polyhedral cone dual to cone spanned by $St(p)$.

Lem: $\exists!$ closest point $\chi \in St(p)$ to origin, and $\nu(p,\lambda)$ is minimized by λ dual to this χ

Pf: homework

Not only is this true for T but there is a unique test datum (x,λ) up to equivalence maximizing $\nu(x,\lambda)$ (recall maps $\mathbb{A}^1/\mathbb{G}_m \to X/G$)

Thm: X projective-over-affine over k, G reductive, $\mathcal{L}$ is NEF class in $NS_G(X)$, then

1) $\forall p \in X^{us}$, $\exists!$ map $f: \mathbb{A}^1/\mathbb{G}_m \longrightarrow X/G$ with an iso $f(1) \cong p$ which maximizes $\nu(f) = \nu(p,\lambda)$, define $M(p) = \nu(f_{max})$

2) if $p \rightsquigarrow q$ then $M(q) \leq M(p)$

3) up to conjugation, only finitely many λ appear as optimal destabilizers.

We will prove this in the case X affine (this implies it when $\mathcal{L}$ is very ample too)

Idea uses spherical building $\mathrm{Sph}(G)$ constructed as follows (recall we are fixing norm $|\bullet|$)

1) $\forall$ max'l tori $T \subset G$, let S_T denote unit sphere in $\mathrm{Lie}(T)_{\mathbb{R}}$

2) Any Borel $B \supset T$ gives a top dim'l cone in $\mathrm{Lie}(T)_{\mathbb{R}}$ (Weyl chamber) $\rightsquigarrow$ gives polyhedral sector $\Delta_B \subset S_T$

3) Glue S_T to $S_{T'}$ along Δ_B if $T, T' \subset B$

<u>Key properties:</u>

$\mathrm{Sph}(G)$ is a union of Δ_B as B ranges over all Borels, intersecting along Δ_P where $B \subset P \supset B'$. Dominant $\lambda : \mathbb{G}_m \to P$ up to conjugation and positive scaling gives point of Δ_P

The function $\nu(p,\lambda) = \dfrac{wt_\lambda \mathcal{L}|_{\lim_{t\to 0} \lambda(t)\cdot x}}{|\lambda|}$ extends to a continuous function

$$\nu : \mathrm{Deg}(p) \longrightarrow \mathbb{R}$$

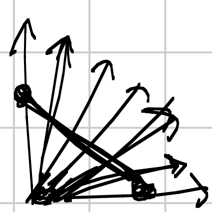
Kempf's theorem says $\exists$ unique minimizer

Pf: 1) existence easy: can reduce to case of $G = T$, because any $(p,\lambda) \sim (gp, g\lambda g^{-1})$ with $g\lambda g^{-1} \in T$

uniqueness: Can define $\mathrm{Deg}(p) \subset \mathrm{Sph}(G)$ to be the closed set of λ s.t. $\lim_{t\to 0} \lambda(t)\cdot p$ exists. For a homom. w/ finite kernel $\mathbb{G}_m^2 \longrightarrow G$ and an equiv. map

$$\mathbb{A}^2/\mathbb{G}_m^2 \longrightarrow X/G \quad \text{mapping } (1,1) \mapsto x$$

get a line segment in $\mathrm{Deg}(p)$



$\leftarrow$ toric diagram for $\mathbb{A}^2$, all lattice points correspond to $\lambda : \mathbb{G}_m \to G$ under which $\lim_{t\to 0} \lambda(t)\cdot p$ exists

Lem: Give two test data (x,λ), (x,λ') can find equivalent test data for x s.t. λ and λ' commute

Pf: if P_1, P_2 are two parabolics, then $\exists$ $T \subset P_1 \cap P_2$

Def: say $\delta, \delta' \in Sph(G)$ are antipodal if $\exists \lambda$ with $\delta = [\lambda]$ and $\delta' = [\lambda^{-1}]$

Lem: Given $\delta, \delta' \in Deg(p) \subset Sph(G)$, (uses X affine!) as long as they are not antipodal, $\exists$ equivariant map

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & X \\ \circlearrowleft & & \circlearrowleft \\ \phi : \mathbb{G}_m^2 & \longrightarrow & G \end{array}$$

mapping $(1,1) \mapsto p \in X$

finite kernel

such that $\delta = [\phi(1,t)]$ and $\delta' = [\phi(t,1)]$

Proof of 2: uses fact that $Y_\lambda \hookrightarrow X$ is a closed immersion, so

$$\begin{tikzcd}
Y_\lambda / P_\lambda \arrow[r] \arrow[d, hook, "\text{cl. imm.}"'] & X/G \\
X / P_\lambda \arrow[ur, "G/P_\lambda\text{-bundle}"'] &
\end{tikzcd}$$

is <u>proper</u> and every point in the image is destabilized by a 1PS conjugate to λ

<u>Proof of 3:</u>